ANALYSIS OF DYNAMICAL SYSTEMS USING LOW SEPARATION AXIOMS (Advances in General Topology and their Problems)

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ABSTRACT. In this paper, we analyze dynamical systems using low separation axioms. In particular, we characterize the T_2 separation axiom for dynamical systems and describe "T_2" dynamical systems. We also characterize recurrence of orbits.

1. INTRODUCTION

In 1927, Birkhoff introduced the concepts of non-wandering points and recurrent points [3]. Using these concepts, we can describe and capture sustained or stationary dynamical behaviors and conservative dynamics. In [2] and [7], it is showed that the following properties are equivalent for an group-action of a finitely generated group G on either a compact zero-dimensional space or a graph X: 1) the group-action is pointwise recurrent; 2) the group-action is pointwise almost periodic; 3) the group-action is R-closed. Since each dynamical system whose orbit class space is T_1 consists of minimal sets, the orbit class spaces of dynamical systems is not T_1 in general. Because the orbit class space of a dynamical system is the T_0-tification of the orbit space, the separation axioms between T_0 and T_1 are important to describe and analyze dynamical systems in detail. Note that higher separation axiom cannot be characterized by the specialization partial order, because the T_1 separation axiom is characterized as an antichain (i.e. a poset where any two distinct elements are incomparable) by the specialization partial order.

2. PRELININAIRES

2.1. **Topological notions.** Define the class $\hat{x}$ of a point x of a topological space (X, τ) by $\hat{x} := \{y \in X \mid \overline{x} = \overline{y}\}$, where $\overline{x}$ is the closure of the singleton $\{x\}$. The quotient space of X by the classes is denoted by $\hat{X}$ (i.e. $\hat{X} := \{\hat{x} \mid x \in X\}$) and called the class space of X. The quotient topology is denoted by $\hat{\tau}$. In other words, the class space $\hat{X}$ of X is the quotient space $X/\sim$ defined by the following relation: $x \sim y$ if $\overline{x} = \overline{y}$.

2.2. **Separation axioms for points.** Let (X, τ) be a topological space. A point $x \in X$ is T_0 if for any point $y \in X - \{x\}$, there is an open subset U of such that $\{x, y\} \cap U$ is a singleton. A point x is T_1 if the singleton $\{x\}$ is closed. For any σ, a point x in X is S_σ if the point $\hat{x}$ in $\hat{X}$ is T_σ. For instance, a point $x \in X$ is S_1 if and only if $\hat{x}$ is a closed point in $\hat{X}$.

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2.3. **Decompositions of topological spaces.** By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint nonempty subsets of a set M such that $M = \bigsqcup \mathcal{F}$, where $\bigsqcup$ is a disjoint union symbol). A decomposition $\mathcal{F}$ can be identified with a quotient space of M, denoted by $M/\mathcal{F}$ and called the decomposition space of M. A subset of a set is $\mathcal{F}$-invariant (or $\mathcal{F}$-saturated) if it is a union of elements of a decomposition

Let $\mathcal{F}$ be a decomposition of a topological space (M, τ). The quotient topology of a decomposition space $M/\mathcal{F}$ is denoted by $\tau_{M/\mathcal{F}}$. The topology $\{\bigsqcup U \mid U \in \tau_{M/\mathcal{F}}\}$ on M is denoted by $\tau_{\mathcal{F}}$ and called the saturated topology on M of $\mathcal{F}$. The union of the class of $L \in \mathcal{F}$ is denoted by $\hat{L}$ and called the class element of L. The decomposition $\{\hat{L} \mid L \in \mathcal{F}\}$ of M is denoted by $\hat{\mathcal{F}}$ and called the class decomposition. The class space of a decomposition space $M/\mathcal{F}$ is denoted by $M/\hat{\mathcal{F}}$ and called the class decomposition space. Then the class decomposition $\hat{\mathcal{F}}$ also can be identified with the class decomposition space $M/\hat{\mathcal{F}}$. Note that the set of saturations of open subsets is a basis of the saturated topology. In the case that a decomposition is either a foliation or the set of orbits of a group-action, the set of saturations of open subsets is the saturated topology [11].

3. S_2 (resp. T_2) separation axiom for dynamical systems

3.1. **Separation axiom for flows.** Define the specialization pre-order $\leq_\tau$ of a topological space (X, τ) as follows: $x \leq_\tau y$ if $x \in \overline{y}$. By a flow, we mean an $\mathbb{R}$-action on a topological space. Note that the set of orbits of a flow v on a topological space M is a decomposition, denoted by $\mathcal{F}_v$, and the decomposition space is called the orbit space of v and denoted by M/v. Moreover the class decomposition space is called the orbit class space and denoted by $M/\hat{v}$. Let v be a flow on a compact Hausdorff space M. The specialization preorder of v is the specialization pre-order $\leq_{\tau_{M/v}}$ on the orbit space M/v. By definitions, we obtain the following observations.

Remark 1. *The following statements hold.*

1) $M/v : T_0 \iff$ *the specialization preorder of v is a partial order.*
2) $M/v : T_1 \iff$ *v is pointwise periodic.*
3) $M/v : S_1 \iff M/\hat{v} : T_1 \iff$ *v is pointwise almost periodic.*
4) $M/v : S_0 \iff M/\hat{v} : T_0 \iff$ *There are no conditions.*

We consider the following complementary questions.

Question 1.

1) $M/v : T_2 \iff ?$
2) $M/v : S_2 \iff M/\hat{v} : T_2 \iff ?$

3.2. **Characterization of S_2 (resp. T_2) separation axiom.** A decomposition $\mathcal{F}$ is upper semicontinuous (usc) if each element of $\mathcal{F}$ is both closed and compact and for any $L \in \mathcal{F}$ and for any open neighbourhood U of L there is a $\mathcal{F}$-saturated neighbourhood of L contained in U, pointwise almost periodic if each class element of it is closed, and R-closed if the subset $R = \{(x, y) \in M \times M \mid y \in \mathcal{F}(x)\}$ is a closed subset, where $\mathcal{F}(x)$ is the element of $\mathcal{F}$ containing $x \in M$. Recall a point x in X is said to be of characteristic 0 [8] if $\hat{\mathcal{F}}(x) = D(x)$ for any $x \in X$, where $\hat{\mathcal{F}}(x)$ is the element of $\hat{\mathcal{F}}$ containing $x \in M$ and $D(x)$ is its (bilateral) prolongation defined as follows: $D(x) = \{y \in X \mid y_\alpha \in \mathcal{F}(x_\alpha), y_\alpha \to y, \text{ and } x_\alpha \to x \text{ for some nets } (y_\alpha), (x_\alpha) \subseteq X\}$. The decomposition is said to be of characteristic

0 if so is each point of it. An pointwise almost periodic decomposition $\mathcal{F}$ is weakly almost periodic in the sense of Gottschalk W. H. if the saturation $\cup_{x \in A} \overline{L_x}$ of closures of elements for any closed subset A of X is closed. Notice that if $\mathcal{F}$ is pointwise almost periodic then $\hat{\mathcal{F}}$ corresponds to the decomposition of closures of elements of $\mathcal{F}$. Weakly almost periodicity in the sense of Gottschalk implies pointwise almost periodicity by definitions. The S_2-separation axiom for orbit spaces is characterized as follows.

Theorem 3.1. [5, 10] *Let v be a pointwise almost periodic flow of a compact Hausdorff space M. The following are equivalent:*

1) *The orbit class space $M/\hat{v}$ is T_2* (i.e. *M/v is S_2*).
2) *The orbit class decomposition $\hat{\mathcal{F}}_v$ is usc.*
3) *The flow v is R-closed.*
4) *The flow v is weakly almost periodic.*
5) *The flow v is of characteristic* 0.
6) *For any open neighbourhood U of each element $\hat{L} \in \hat{\mathcal{F}}_v$, there is an open $\hat{\mathcal{F}}_v$-saturated neighbourhood of $\hat{L}$ contained in U.*

This implies the following characterization of the Hausdorff separation axiom.

Corollary 3.2. *Let v be a pointwise periodic flow of a compact Hausdorff space M. The following are equivalent:*

1) *The orbit space M/v is T_2.*
2) *The orbit decomposition $\mathcal{F}_v$ is usc.*
3) *The flow v is R-closed.*
4) *The flow v is weakly almost periodic.*
5) *The flow v is of characteristic* 0.

The previous theorem is followed from the key lemma.

Lemma 3.3. [5, 10] *Let $\mathcal{F}$ be a pointwise almost periodic decomposition of a compact Hausdorff space X which consists of connected elements. The following are equivalent:*

1) *The decomposition $\mathcal{F}$ is R-closed.*
2) *The decomposition $\mathcal{F}$ is weakly almost periodic.*
3) *The decomposition $\mathcal{F}$ is of characteristic* 0.
4) *The class decomposition $\hat{\mathcal{F}}$ is T_2* (i.e. *$\mathcal{F}$ is S_2*).
5) *The class decomposition $\hat{\mathcal{F}}$ is usc.*
6) *For any open neighbourhood U of each element $\hat{L} \in \hat{\mathcal{F}}$, there is an open $\hat{\mathcal{F}}$-saturated neighbourhood of $\hat{L}$ contained in U.*

3.3. **T_2 separation axiom for flows on compact 3-manifolds.** Recall that a point x of S is singular if $x = v_t(x)$ for any $t \in \mathbb{R}$, is regular if x is not singular, and is periodic if there is a positive number $T > 0$ such that $x = v_T(x)$ and $x \neq v_t(x)$ for any $t \in (0, T)$. Denote by $\mathrm{Sing}(v)$ the set of singular points and by $\mathrm{Per}(v)$ the union of periodic orbits. By a continuum we mean a compact connected metrizable space. A continuum $A \subset X$ is said to be annular if it has a neighborhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components, both homeomorphic to annuli. A subset $C \subset X$ is a circloid if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset. We state the following trichotomy that an R-closed flow on a connected compact

3-manifold is either "almost three dimensional", "almost two dimensional", "almost one dimensional", "almost zero dimensional", or with "complicated" minimal sets.

Theorem 3.4. [10] *Let v be an R-closed flow on a connected compact 3-manifold M. Then one of the following holds:*
1) the flow v is identical.
2) the flow v is minimal.
3) The orbit class space $M/\hat{v}$ of M is a closed interval or a circle and each interior point of the orbit class is two dimensional.
4) $\mathrm{Per}(v)$ *is open dense and* $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v)$.
5) There is a two dimensional minimal set which is not a suspension of a circloid.

3.4. T_2 **separation axiom for "Codimensionone (resp. two) like" group-actions.** By a group-action, we mean a continuous action of a topological group on a topological space. For a non-negative integer k, a group-action G is said to be codimension-k-like if all but finitely many orbit closures of $\mathcal{F}_G$ are codimension k connected submanifolds without boundaries and the finite exceptions is connected subsets each of whose codimension is more than k, where $\mathcal{F}_G$ is the set of orbits of G. We have the following results.

Theorem 3.5. [10] *The orbit class space of an R-closed group-action on a compact connected manifold one of whose finite index subgroups is codimension-one-like is either a closed interval or a circle.*

Theorem 3.6. [10] *The orbit class space of an R-closed group-action on a compact connected manifold one of whose finite index subgroups is codimension-two-like is a surface with corners.*

4. Topological characterization of recurrence by separation axioms

4.1. T_1 **(resp.** S_1**) separation axiom and Minimality for decompositions.** For a decomposition $\mathcal{F}$ on a set M, a nonempty closed $\mathcal{F}$-invariant subset of a topological space is a $\mathcal{F}$-minimal set (or $\mathcal{F}$-minimal) if it there are no nonempty closed $\mathcal{F}$-invariant proper subset of it. A point x of a topological space is C_R [9] if the derived set $\overline{x} - \{x\}$ contains no nonempty closed subsets. We have following observations.

Lemma 4.1. [12] *The following statements are equivalent for an element O of a decomposition $\mathcal{F}$ on a topological space:*
1) *O is T_1.*
2) *O is $\mathcal{F}$-minimal.*
3) *O is minimal in $M/\mathcal{F}$ with respect to the specialization preorder* (i.e. $O \in \min M/\mathcal{F}$) *and $O = \hat{O}$.*

Lemma 4.2. [12] *The following statements are equivalent for an element O of a decomposition $\mathcal{F}$ on a topological space:*
1) *O is S_1.*
2) *O is C_R.*
3) *$\overline{O}$ is $\mathcal{F}$-minimal.*
4) $\overline{O} = \hat{O}$.
5) *$\hat{O}$ is minimal in $M/\hat{\mathcal{F}}$ with respect to the specialization preorder* (i.e. $\hat{O} \in \min M/\hat{\mathcal{F}}$).

Note that the condition “O is C_R” means that the derived set $\overline{O} - O$ contains no nonempty $\mathcal{F}$-invariant closed subsets.

4.2. **Propeness for topological spaces.** A point x of a topological space X is proper if there is its neighborhood U in which x is closed (i.e. $\overline{x} \cap U = \{x\}$). A point $x \in X$ is T_D [1] if the derived set $\overline{x} - \{x\}$ is a closed subset. Obviously we have the following equivalence.

Lemma 4.3. [12] *A point of a topological space is proper if and only if it is T_D.*

For orbits of flows on manifolds, properness corresponds to T_0 separation axiom. Precisely, the following statement is follows from Cherry's technique essentially[4].

Lemma 4.4. [12] *The following statements are equivalent for an orbit O of a flow on a paracompact manifold:*
1) *O is proper.*
2) *O is T_D.*
3) *O is T_0 (i.e. $O = \hat{O}$).*

In [9], a point $x \in X$ is C_D if the derived set $\overline{x} - \{x\}$ of x is either empty or non-closed, and it is C_0 if the derived set $\overline{x} - \{x\}$ is not a union of nonempty closed subsets. These axioms satisfies the following relations [9]: $S_1 \Rightarrow C_0 \Rightarrow C_D$. We have the following characterization of C_0 and C_D by using pre-order.

Lemma 4.5. [11] *Let x be a point of a topological space X. The following statement holds:*
1) *x is C_0 $\iff$ $x \in \min X$ or $|\hat{x}| > 1$.*
2) *x is C_D $\iff$ $x \in \min X$ or $x \in \overline{\overline{x} - \{x\}}$.*

We can summarize the following topological characterization of recurrence.

Theorem 4.6. [11] *Let v be a flow on a compact metrizable space M and O an orbit of v. The following statements are equivalent for the orbit space M/v:*
1) *O is recurrent*
2) *O is either T_1 or non-T_D* (i.e. *O is closed or non-proper*).
3) *O is either S_1 or non-T_D* (i.e. *O is minimal or non-proper*).
4) *O is either C_R or non-T_D.*
5) *O is C_D.*

Moreover, if M is a manifold, then the following conditions are equivalent to any of above conditions:
6) *O is either T_1 or non-T_0.*
7) *O is either S_1 or non-T_0.*
8) *O is either C_R or non-T_0.*
9) *O is C_0.*

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